Scheduling using max-algebra

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$$a \oplus b = \max(a, b)$$
$$a \otimes b = a + b$$
$$a, b \in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\}$$

Basic properties ($\varepsilon = -\infty$, $a^{-1} = -a$):

$$a \oplus b = b \oplus a$$
$$(a \oplus b) \oplus c = a \oplus (b \oplus c)$$
$$a \oplus \varepsilon = a = \varepsilon \oplus a$$

$$a \otimes b = b \otimes a$$
$$(a \otimes b) \otimes c = a \otimes (b \otimes c)$$
$$a \otimes \varepsilon = \varepsilon = \varepsilon \otimes a$$
$$a \otimes 0 = a = 0 \otimes a$$
$$a \otimes a^{-1} = 0 = a^{-1} \otimes a$$

$$(a \oplus b) \otimes c = a \otimes c \oplus b \otimes c$$
$$a \oplus b = a \text{ or } b$$

Extension to matrices and vectors:

$$A \oplus B = (a_{ij} \oplus b_{ij})$$
$$A \otimes B = \left(\sum\nolimits_k^{\oplus} a_{ik} \otimes b_{kj}\right)$$
$$\alpha \otimes A = (\alpha \otimes a_{ij})$$

$$A \oplus B = B \oplus A \qquad [\text{not } A \otimes B = B \otimes A]$$

$$(A \oplus B) \oplus C = A \oplus (B \oplus C) \qquad (A \otimes B) \otimes C = A \otimes (B \otimes C)$$

$$A \otimes \varepsilon = \varepsilon = \varepsilon \otimes A \qquad A \otimes I = A = I \otimes A$$

$$(A \oplus B) \otimes C = A \otimes C \oplus B \otimes C$$

$$A \otimes (B \oplus C) = A \otimes B \oplus A \otimes C$$

A^{-1} exists $\Longleftrightarrow$ A is a permutation matrix ($=$ permuted diag)

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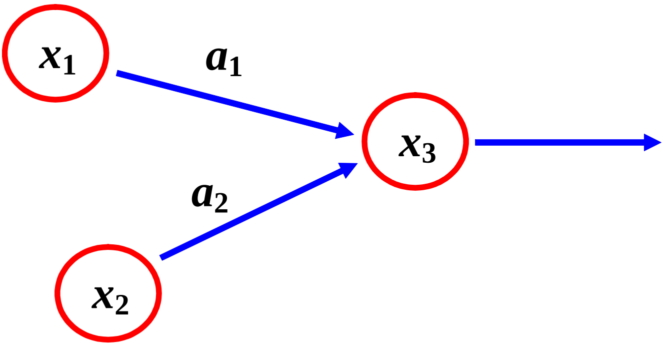
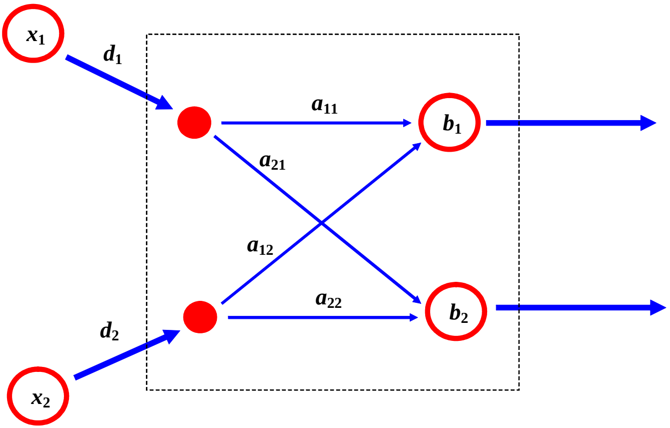
- $(A \oplus B)^k \neq A^k \oplus B^k$
- $(I \oplus A)^k = I \oplus A \oplus A^2 \oplus ... \oplus A^k$



$$
\begin{aligned}
x_3 &= \max(x_1 + a_1, x_2 + a_2) \\
&= a_1 \otimes x_1 \oplus a_2 \otimes x_2 \\
&= (a_1, a_2) \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = a^T \otimes x
\end{aligned}
$$



$$b_1 = \max(x_1 + d_1 + a_{11}, x_2 + d_2 + a_{12})$$
$$b_2 = \max(x_1 + d_1 + a_{21}, x_2 + d_2 + a_{22})$$

$$b = A \otimes x$$

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- If no solution then $A \otimes x \leq b$ but as tight as possible

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- $\overline{x} = A^* \otimes' b$, where $A^* = -A^T$ and $\otimes'$ is in min-algebra
- Then $A \otimes \overline{x} \leq b$ and $A \otimes \overline{x}$ is the best Chebyshev approximation of b

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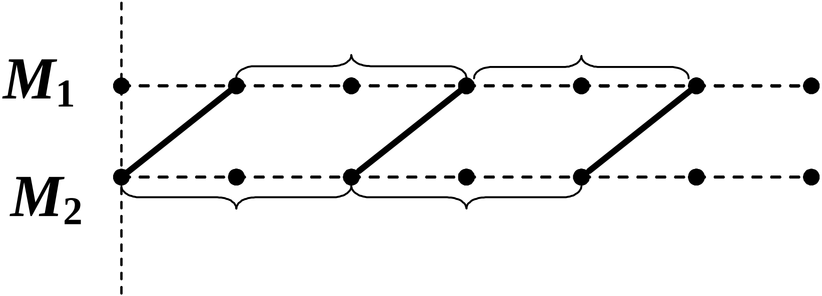
- $A : x(0) \rightarrow x(1) \rightarrow x(2) \rightarrow ...$

$$A : x(0) \longrightarrow x(1) \longrightarrow x(2) \longrightarrow \ldots$$

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} : \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 3 \\ 2 \end{pmatrix} \to \begin{pmatrix} 5 \\ 4 \end{pmatrix} \to \begin{pmatrix} 7 \\ 6 \end{pmatrix} \to \ldots$$



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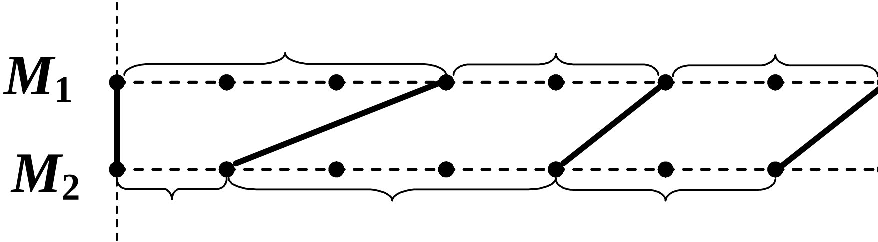
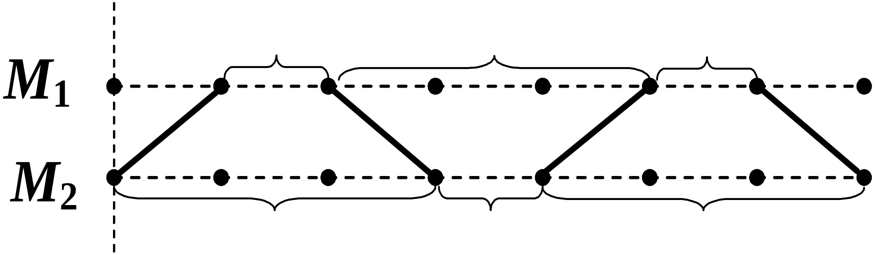


$\lambda = 2, r_0 = 2$

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} : \begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 2 \\ 3 \end{pmatrix} \rightarrow \begin{pmatrix} 5 \\ 4 \end{pmatrix} \rightarrow \begin{pmatrix} 6 \\ 7 \end{pmatrix} \rightarrow \begin{pmatrix} 9 \\ 8 \end{pmatrix} \rightarrow \dots$$



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- Since

$$x(r) = A \otimes x(r-1) = A^{(2)} \otimes x(r-2) = \ldots \; = A^{(r)} \otimes x(0),$$

a steady regime is reached if and only if $A^{(r)} \otimes x(0)$ "hits" an eigenvector of A for some r.

Given: $A = (a_{ij}) \in \overline{\mathbb{R}}^{n \times n}$

- **Problem 1:** Find a $\lambda \in \mathbb{R}$ (*eigenvalue*) and an $x \in \overline{\mathbb{R}}^{n}$ (*eigenvector*) so that

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- **Problem 3:** Is A *robust*? (That is for **every** $x \in \overline{\mathbb{R}}^n$, $x \neq \varepsilon$, there is a k such that $A^{(k)} \otimes x$ is an eigenvector of A)

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  associated digraph

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- The *maximum cycle mean of* A :

$$\lambda(A) = \max\left\{\frac{a_{i_1 i_2} + a_{i_2 i_3} + \ldots + a_{i_k i_1}}{k}; (i_1, \ldots, i_k, i_1) \text{ cycle in } D_A\right\}$$

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then $\lambda(A) = \max\{-2, 0, 1, 2, 1, 5/2, 3, 2/3\} = 3$

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- $\Gamma(A) = A \oplus A^2 \oplus \ldots \oplus A^n$ (*metric matrix*)

Theorem (R.A.Cuninghame-Green)

If $A \in \overline{\mathbb{R}}^{n \times n}$ is irreducible then $\lambda(A)$ is the unique eigenvalue of A and the columns of $\Gamma\left((\lambda(A))^{-1} \otimes A\right)$ with zero diagonal entries are a complete set of generators of the eigenspace of A.

Theorem (G.Cohen, D.Dubois, J.-P.Quadrat, M.Viot)

(Cyclicity Theorem) If $A \in \overline{\mathbb{R}}^{n \times n}$ *is irreducible, then A is ultimately periodic, that is there exist positive integers p and* k_0 *such that:*

$$A^{(k+p)} = (\lambda(A))^{(p)} \otimes A^{(k)} \text{ for all } k \geq k_0.$$

p smallest ... p is the *period* of A, $p = p(A)$

Theorem

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Theorem

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- *A is robust*
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- *Eigenspaces of all powers of A coincide.*

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- *Max-linear programming* ...pseudopolynomial algorithm [PB+AA]

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s.t.

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- *Optimizing starting times* ... polynomial/NP-complete [PB+KPT]

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- $M_2 \cup M_3 = M$ hence
$$S(A, b) = \left\{ (x_1, x_2, x_3)^T \in \overline{\mathbb{R}}^3 ; x_1 \leq 3, x_2 = 1, x_3 = -2 \right\}.$$

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Theorem (S.Gaubert, 1992)

(Spectral Theorem) If $A \in \overline{\mathbb{R}}^{n \times n}$ *then*

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- N_1, N_4 are spectral

N_i is called *trivial* if A_{ii} is 1×1 matrix (ε)

Theorem

(Robustness criterion) If $A \in \overline{\mathbb{R}}^{n \times n}$, $A \neq \varepsilon$ then A is robust if and only if

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- $p(A_{ii}) = 1 \quad (i \in R)$
- $\lambda(A_{ii}) = \lambda(A_{jj})$ *if* N_i, N_j *are non-trivial,* $N_i \nrightarrow N_j$ *and* $N_j \nrightarrow N_i \quad (i, j \in R)$

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- The reason: $N_1 \nrightarrow N_2$ and $N_2 \nrightarrow N_1$ but $\lambda(N_1) \neq \lambda(N_2)$.

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R.A.Cuninghame-Green 1960
N.N.Vorobyov 1967, Litvinov, Maslov, Kolokoltsov, Sobolevski...
M.Gondran, M.Minoux 1975
K.Zimmermann 1972
P.Butkovič 1977
R.E.Burkard, U.Zimmermann 1981
H.Schneider
G.Cohen, D.Dubois, J.-P.Quadrat, M.Viot 1983
K.Cechlárová, P.Szabó, J.Plávka 1985
G.-J.Olsder, C.Roos 1988, B.Heidergott 2000
R.D.Nussbaum 1991
M.Akian 1998, S.Gaubert 1992, C.Walsh 2001
R.B.Bapat, D.Stanford, P.van den Driessche 1993
M.Gavalec 1995
Tropical algebra from 1995: B.Sturmfels, M.Develin et al